A FAMILY OF DISTRIBUTIONS TO DESCRIBE A MULTIPROJECTILE ROUND

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THESIS

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to Describe a Multiprojectile Round

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ABSTRACT

This paper is an analysis of the behavior of density shot XM261 when fired from the .45 caliber pistol (M1911A1). Density shot ammunition is a multiprojected round which contains 16 small pellets. Both the standard rifled barrel and a smooth barrel were used to collect data on the behavior of the XM261 round. A family of distributions is selected for each barrel type to model the scatter of pellets from the XM261 round at varying ranges.

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I. INTRODUCTION

The United States small arms systems agency (USASASA) is investigating three different types of ammunition for use with the .45 caliber pistol (M1911A1). The three types of ammunition being considered are the standard ball ammunition, a segmented projectile which breaks into six pieces after leaving the barrel, and the density shot XM261 which contains 16 small pellets. USASASA is also investigating the effects that a smooth bore barrel and the standard rifled barrel have on each type of ammunition. USASASA is trying to determine if greater effectiveness can be achieved with the .45 pistol by changing the combination of ammunition and barrel. This paper will deal only with the density shot ammunition. H. P. White Laboratory, Bel Air, Maryland, was contracted to provide velocity and accuracy data for the evaluation of the six combinations of ammunition and barrel types. In February 1973, the laboratory published a final report containing the data required [Ref. 1].

The purpose of this paper is to document an analysis of that segment of the data provided by the H. P. White Laboratory that concerns the XM261 ammunition. Data was originally taken at three ranges: 10, 30, and 50 meters. However, so many pellets missed the target paper at 50 meters that the data did not lend itself to analysis at that range. Consequently, the analysis was conducted with only four sets of

data: 10 and 30 meter ranges with a smooth barrel and 10 and 30 meters with a rifled barrel.

The problem posed by USASASA was to determine a bivariate distribution which could be used to model the scatter of pellets about the aiming point. This distribution could be used, for example, in computer simulations to determine the effectiveness of the .45 caliber pistol using the density shot round. Because of a projected use of the distribution in a computer simulation, USASASA desired that the form of the distribution lend itself to an easy generation of data within the context of a computer simulation.

The analysis identified a family of distributions which can be used to model the scatter of the pellets about the center of mass of a single shot. The family of distributions utilizes the polar coordinate method of describing the data. The angle is distributed uniformly from zero to two pi, the radius vector is distributed according to a gamma distribution, and the angle and radius vector are independent. The parameters for the gamma distribution are estimated from the data.

II. DATA COLLECTION

The data analyzed in this paper was collected by H. P. White Laboratory under contract DAA05-73-M-2224 [Ref. 1]. The procedure used to obtain the data is explained in the following excerpt from the H. P. White report:

PREPARATION

The model 1911A1 pistol was mounted in the Broadway machine rest and boresighted at a reference point on an 8'X8' paper target at 50 meters. Paper targets 6'X6' and 4'x4' were then erected at 30 and 10 meters, respectively, with reference point in the line of the bore sighting. Velocity screens were erected at 4 and 25 feet from the muzzle to record velocity at 15 feet from the muzzle.

DATA COLLECTION

Velocity was recorded for each round (leading projectile only of multiprojectile rounds). The impact of each projectile on each of the three (3) targets (10, 30, and 50 meters) was recorded in relation to the line of the bore.... Since each XM261 round contains 16 pellets, the data provided by H. P. White normally consisted of 160 impact points for each of the three ranges used. However, more than 40 percent of the pellets fired from the rifled barrel failed to impact on the 8'X8' target at 50 meters. Consequently, this student concluded that the maximum effective range of the XM261 round fired from a rifled barrel was less than 50 meters. Therefore,

no attempt was made to analyze the 50 meter data. This left four sets of data to analyze: 10 meter range with a rifled barrel, 30 meter range with a rifled barrel, 10 meter range with a smooth barrel, and 30 meter range with a smooth barrel. Each set of data consisted of 160 sets of X and Y coordinates: ten shots with 16 pairs of coordinates for each shot. The data is listed in Appendix A.

III. SELECTION OF DISTRIBUTIONS

For the present analysis, the basis for selecting distributions to model the scatter of pellet strikes from the XM261 was a letter from the United States Army Material Systems Analysis Agency (USAMSAA) [Ref. 2]. USAMSAA was replying to a request from USASASA to analyze data obtained from firing the XM261 round. The data analyzed by USAMSAA was prepared by the Frankford Arsenal. As a result of their analysis, USAMSAA suggested that the distribution of pellet strikes from the XM261 round when fired from a rifled barrel could be modeled by using a polar coordinate system with the angle having a uniform distribution, the radius vector having a normal distribution, and the angle and radius vector being independent. (The model based on these assumptions will hereafter be called the Polar Model.) When the XM261 round was fired from a smooth bore barrel, USAMSAA suggested that a bivariate normal distribution with independent X and Y would serve as an adequate model. This student felt that these two models could serve as an adequate basis for an initial investigation of the XM261 data provided by H. P. White Laboratory.

The plane had to be partitioned into regions so that a chi-square goodness-of-fit test could be performed. The objective for the partition was to allow the maximum number of regions consistent with the requirement to have an expected frequency of observation of pellets of approximately five [Ref. 3 and 4]. In connection with testing the fit of the

Polar Model, the plane was originally divided into 30 nearly equally probable regions. Since the angle was assumed to have a uniform distribution, the plane was first divided into six regions by constructing six rays from the origin at 60 degree intervals. Then four circles were superimposed over the rays to further divide the plane. The radii of the circles corresponded to the 20, 40, 60, and 80 percentiles of the fitted distribution of the radius vector. This resulted in a partition of the plane into 30 nearly equally probable regions (see Figure 1). Since there were 160 data points, the expected frequency for each of the regions was 5.33.

When testing the fit of the bivariate normal distribution, the plane was divided into 36 rectangles. The mean and standard deviation of the X variable were estimated using the maximum likelihood estimates. Then values for the 16.67, 33.33, 50, 66.67, and 83.33 percentiles of the X variable were estimated using the estimates for the mean and standard deviation. A vertical line was constructed on the X value which corresponded to each of these estimated percentiles. Then using the same method horizontal lines were constructed on the same sample percentiles of the Y variable. The resulting partition provided 36 nearly equally probable rectangles (see Figure 2). The expected frequency based on the 160 data points was 4.45. Although this was less than the desired frequency of five, it was sufficient [Ref.3].

PARTITION OF THE PLANE FOR THE POLAR DISTRIBUTION

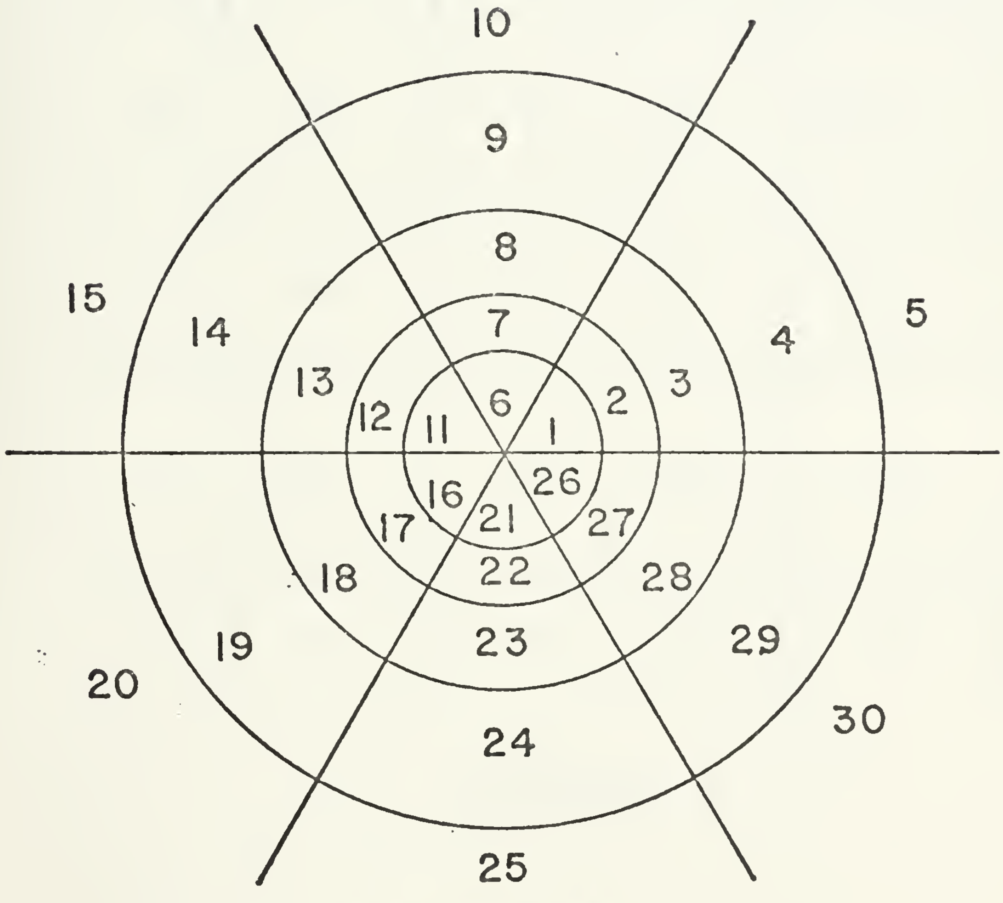

FIGURE 1

PARTITION OF THE PLANE FOR THE BIVARIATE NORMAL

31	32	33	34	35	36
25	26	27	28	29	30
19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

FIGURE 2

IV. ANALYSIS OF DATA FROM THE RIFLED BARREL

A. INITIAL RESULTS

When fitting the bivariate normal distribution, four parameters, the mean and standard deviation of both the X and Y variables, had to be estimated. Since the X and Y variables were assumed to be independent, there was no need to estimate a correlation coefficient. Because there were 36 regions and four parameters estimated, this chi-square test had 31 degrees of freedom. This yielded a critical value of 45.0 for a five percent level of significance and a critical value of 41.5 for a ten percent level of significance. The test statistic had a value of 95.6 for the 10 meter data and a value of 93.8 for the 30 meter data. Hence, the hypothesis that either set of data was distributed according to a bivarate normal was rejected.

When fitting the Polar Model, only two parameters, the mean and standard deviation of the distribution of the radius vector, were estimated. Because the Polar Model had 30 regions and two parameters were estimated, this chi-square statistic had 27 degrees of freedom. The critical values for five and ten percent levels of significance are 40.1 and 36.7, respectively. The test statistic for the 10 meter data was 55.25, that for the 30 meter data was also 55.25. This led to the rejection of the hypothesis that either set of data was distributed according to the Polar Model.

Since both distributions had been rejected as methods for adequately describing the data, an additional distribution was sought. A histogram of the values for the radius vector suggested that the radius vector might be distributed according to a gamma distribution. Therefore, an additional distribution was introduced. This distribution utilized the polar coordinate description of the data with the assumptions that the angle was uniformly distributed between zero and two pi, the radius vector was distributed according to a gamma distribution, and the angle and the radius vector were independent. This distribution will be referred to as the Polar Gamma distribution. The plane was partitioned for the chi-square goodness-of-fit test in the same way as for the Polar Model. The values of the radii of the circles corresponded to the appropriate percentiles of the gamma distribution of the radius vector.

If a gamma distribution was to be used to represent the distribution of the radius vector, the parameters would have to be estimated from the data. The maximum liklihood method of estimating the parameters has some appeal, but it is cumbersome because it involves the gamma function. Therefore, it was decided to use the method of moments to estimate the gamma parameters. This decision was made in spite of the fact that the method of moments produced a biased estimate of the parameters. Appendix B contains the method of moments estimates and a justification that they are biased. Even after the parameters were estimated some

difficulty is experienced in estimating the 20, 40, 60, and 80 percentiles of a general gamma distribution to use a values for the radii of the circles separating the different regions in the plane. However, if the shape parameter alpha is a positive integer, then a convenient relationship exists between the gamma and the Poisson distributions. Specifically, if X is distributed gamma (M,λ) and Y is distributed Poisson $(\lambda\chi)$, then $P(X \leq \chi)=P(Y \geq M)$ [Ref. 5]. Therefore, the percentile values for the gamma distribution with an integer alpha value can be interpolated from a table of cumulative values for the Poisson distribution.

One of the considerations for this problem was that the distribution used to describe the data should lend itself to easy generation by computer simulation. Hence, it was decided that after the gamma parameters were estimated using the method of moments, the integers on either side of the estimated alpha value would be used instead of the estimated alpha value. This approach would permit the use of the relationship between the gamma and the Poisson distributions. Each data set would then yield two chi-square test statistics for each set of gamma parameters estimated: One test would use the integer value immediately below the estimated alpha value, the second would use the integer value immediately above the estimated alpha value. Both tests would use the estimated lambda value.

Since fitting the polar gamma also requires the estimation of two parameters and the plane had been partitioned into 30

regions, the chi-square goodness-of-fit test for the Polar Gamma had 27 degrees of freedom. Consequently, the critical values for the test statistic were the same as for the Polar Model. The values for the two test statistics generated using the 10 meter data with the rifled barrel were 62.0 and 48.1. Those for the 30 meter data with the rifled barrel were 58.6 and 52.6. Therefore, both of the null hypotheses that the two sets of data were distributed according to the Polar Gamma distribution were rejected. Although the bivariate normal, Polar, and Polar Gamma distributions had been rejected as models for the data, the Polar Gamma distribution had produced the lowest value for the test statistic for both the 10 and 30 meter data. This indicated that the Polar Gamma distribution might have some potential to describe the data.

Because all the distributions had failed to produce an acceptable description of the data, a check was made to see if the data contained any identifiable variation that had not been accounted for. The physical characteristics of the method of data collection led to the belief that such a variation might exist. The recoil and rifling of the barrel were suspected of causing the center of mass (sample mean) of a pellet pattern to drift up and to the left as the range increased. The recoil was suspected of causing the drift up. The movement of the barrel of one half of one degree is sufficient to cause the center of mass to move more than 10 inches at 30 meters. The rifling was

suspected of causing the drift to the left. Since the left handed rifling produces a counterclockwise rotation of the projectile, the projectile would tend to drift to the left when meeting air resistance. If the center of mass did in fact drift from the aim point and if the drift was random, then this variation in the location of the centers of mass would introduce an uncontrolled variation among the rounds. This would be considered undesirable since the chi-square goodness-of-fit test assumes that all observations (shots) come from the same distribution.

In order to determine whether the data would tend to support or refute the hypothesis that the centers of mass did drift as range increased, the center of mass of each shot was calculated from the data at ranges of 10 and 30 meters. Each center of mass was based on the 16 pellets in one XM261 round. An examination of the centers of mass calculated in this fashion tended to support the hypothesis that the center of mass drifted up and to the left as range increased. For both the 10 and 30 meter data eight out of ten X coordinates of the centers of mass were negative and nine out of ten Y coordinates were positive. Furthermore, a comparison of the centers of mass at 10 and 30 meters for the same round revealed that eight out of ten centers of mass had drifted up and to the left. It was also noted that the actual distance that the center of mass moved varied significantly among the rounds. The calculated centers of mass are provided in table 1.

TABLE I

CENTERS OF MASS-RIFLED BARREL

Round	10 Meters	30 Meters
1	(0.25,1.14)	(-0.46,2.25)
2	(-0.10,0.76)	(-1.27,2.80)
3	(-0.41,2.88)	(0.50,8.35)
4	(0.00,1.36)	(-3.03,2.83)
5	(-1.92,-1.31)	(-3.76,-4.30)
6	(-0.43,2.33)	(-8.43,3.92)
7	(-0.85,0.64)	(-1.45,2.15)
8	(-0.53,1.48)	(-0.89,2.51)
9	(-2.50,2.27)	(-5.37,2.58)
10	(1.29,0.05)	(-4.08,0.56)
Overall Mean	(-0.52,1.16)	(-2.82,2.36)

In order to eliminate the apparently random drift of the centers of mass, the coordinates for each group of 16 pellets from a single shot were modified by subtracting the X and Y coordinates of that shot's center of mass from the X and Y coordinates of each pellet in that group. This made each group of pellets with modified coordinates have a center of mass at the origin.

An interesting phenomenon may occur when each group of pellets has a center of mass at the origin. The data may appear to have an area of relatively low probability at the origin (i.e., a 'hole' may appear in the data). This may be true even though the original data may not have exhibited this trait. A possible explanation for the 'hole' appearing is that when the centers of mass are not constrained to be at the origin, the area of low probability for one shot may lie in an area of high probability for a different shot. The overlap of the different patterns from the different rounds may prevent a 'hole' from being observed. However, when the centers of mass are all at the origin, the areas of low probability may all coincide to produce a 'hole' in the data.

Another phenomenon may produce a 'hole' in the data even when the distribution of the data does not exhibit an area of low probability. For example, if one drew pairs of numbers from a uniform distribution and plotted the distance of each point from the center of mass of its pair of points, then he would observe an area of low probability at the

origin. However, if one simply plotted the points on the unit interval, no such area of low probability would be observed. So it is possible to produce a 'hole' in a data set simply by plotting distances from the center of mass a group of data points. Regardless of which type of phenomenon is occurring, it seems important to keep in mind that the characteristics of the data may be altered when the pellet strikes are measured from the center of mass of each shot.

Although the modification of the data seemed convenient to allow for its analysis, this procedure introduced three significant problems. First, it complicated the problem of utilizing any distribution to measure the effectiveness of the round by USASASA. Any distribution produced by subsequent analysis would reflect the scatter of pellets about the center of mass, but would suppress the drift of the center of mass from the aim point. The drift of the center of mass would have to be reintroduced in any simulation of the actions of this round. Failure to do so might produce erroneous estimates of the effectiveness of this round. Procedures to account for the drift of the center of mass are proposed in Appendix C.

A second problem produced by modifying the data is that when all groups of shots have the same origin, measuring pellet strikes from the center of mass rather than the origin may modify the distributional characteristics of the scatter of the pellets. (This problem is different from the phenomenon discussed above. There this student commented that constraining all pellets to have their centers of mass

at the origin might produce a 'hole' in the data. Here he is considering the effect of measuring coordinates from the center of mass given that the expected value for the center of mass is the origin.) An investigation into this area indicated that, at least for a bivariate normal, the probability that a pellet will strike within a specified distance of the origin is smaller than the probability that a pellet will strike within the same distance of the center of mass. See appendix D for a further discussion of this topic. Since measuring from the center of mass does modify the distribution of the data when a bivariate normal distribution is used, one would suspect that a similar phenomenon might occur with other distributions. Hence, centering the pellets at their respective center of mass may appear to increase the probability of striking close to the origin.

The final problem produced by modifying the data concerns the degree to which the data conforms to the assumptions required for the chi-square coodness-of-fit test after it has been modified. Even if the data had been independent in its original form, it is important to note that after the data has been modified, the coordinates of the pellets in one shot are no longer independent. This is so because all the coordinates of the pellet locations depend on the coordinates of the center of mass. Since the center of mass is a function of all the data points in one group, all of the pellet coordinates in a single shot are statistically

dependent. Distribution of the chi-square statistic assumes independent data points, so modifying the data in this way causes a departure from the chi-square assumptions. If the chi-square distribution is not applicable, then the critical values based on the chi-square distribution may no longer be valid. Because of this, it seemed reasonable to simulate the goodness-of-fit procedure in order to generate appropriate critical values. After performing the chi-square goodness-of-fit test with modified data, the distribution yielding the smallest value was selected as the best candidate to describe the data. Using this best candidate distribution, 1000 sets of data points were generated by a computer. Each set of data was subjected to the goodness-of-fit procedure and the resultant test statistic was stored. The 1000 test statistics were ordered from smallest to largest and several sample percentiles of the test statistic were printed. The values of the 900 and 950 test statistics were used as the critical values to test the hypothesis that the data could be modeled by the test distribution with level of significance of five and ten percent, respectively. The simulation is discussed at appendix E.

B. FINAL RESULTS FOR THE RIFLED BARREL DATA

After modifying the data by subtracting the coordinates of the appropriate centers of mass, the parameters of the gamma distribution for the radius vector were estimated for both the 10 and 30 meter data. These revised parameter estimates were used when the data was compared against the

Polar Gamma distribution. For the 10 meter data the value of the test statistic with the bivariate normal distribution was still high: 100.5. The value of the test statistic for the Polar Model was much lower: 23.0. Moreover, for the 10 meter data the test statistics for the Polar Gamma Model were still lower with values of 21.9 for parameter values of alpha equal to 16 and lambda equal to 1.5766 and 20.7 for parameter values of alpha equal to 17 and lambda equal to 1.5766. Since the critical values based on the simulation were 41.3 and 44.7 for levels of significance of ten and five percent, respectively, the Polar Model and the two Polar Gamma Models could not be rejected.

A similar pattern emerged when the 30 meter data was analyzed. The test statistic with the bivariate normal distribution was 88.4. That for the Polar Model was 30.5. the Polar Gamma distribution with alpha equal to 15 and lambda equal to 0.4977 yielded a test statistic of 30.5. When the gamma parameters were alpha equal to 16 and lambda equal to 0.4977, a test statistic of 27.1 was obtained. Again the Polar Model and the two Polar Gamma Models could not be rejected.

Subjective considerations were used in recommending a distribution, from those not previously rejected, to model the data. It is intuitively appealing to choose the distribution which has the lowest value for the test statistic. Using this criterion for the 10 meter data, both Polar Gamma distributions would be preferred to the Polar

distribution. For the 30 meter data the Polar Gamma distribution with alpha parameter equal to 16 would be preferred above all other distributions. However, when choosing between the two Polar Gamma distribution for the 10 meter data, an additional consideration should be included. It seemed reasonable that the distribution selected to model the scatter of pellets at 10 meters should resemble the 30 meter distribution as closely as possible. The addition of the consideration favors the selection of the Polar Gamma distribution with alpha equal to 16, even though the test statistic for the other Polar Gamma distribution is slightly smaller. Therefore, this student would select the Polar Gamma distribution with alpha equal to 16 as a model for the scatter of pellets fired from a rifled barrel. Additional justification for preferring the Polar Gamma distribution over the Polar distribution is found in the characteristics of the different distribution. Since the Gamma distribution is defined only for positive values, using a Gamma distribution to model the distribution of values of the radius vector is appealing. The normal distribution, on the other hand, is unbounded in both the positive and negative directions. This always allows the theoretical possibility of having a negative value for the radius vector. If the mean of the distribution of the radius vector is not large with respect to the standard deviation, using a normal distribution could produce a significant number of negative values.

Moreover, the histogram of data values for the radius vector appeared to resemble a gamma distribution more than a normal distribution. Finally, the gamma distribution is easily modified to allow for variation in ranges. Only the scale parameter lambda need be modified to reflect a change in range. Possible procedures to account for the change in lambda as the result of a change in range are discussed in appendix F.

V. ANALYSIS OF SMOOTH BARREL DATA

The analysis of the smooth barrel data was similar to the analysis of the rifled barrel data. The parameters for the Gamma distribution were estimated using the method of moments. The three distributions, bivariate normal, Polar, and Polar Gamma, were then used to determine if they could adequately explain the data. All three failed to provide an acceptably small value for the test statistic. For example, the 10 meter data produced test statistics of 145.5, 291.0, 359,2, and 248.5 for the bivariate normal, polar, and the two versions of the Polar Gamma distributions, respectively.

Following the technique of the previous section, the centers of mass were calculated using the data points in each round. These calculated centers of mass were then examined. As with the rifled barrel, the smooth bore data centers exhibited a random drift. However, the characteristics of the drift were slightly different. Since the barrel was no longer rifled and since the rifling was suspected of producing the drift toward the left, one would expect the smooth barrel to fail to consistently produce a movement to the left. The smooth barrel data was, in fact, free from this tendency. Essentially half of the centers of mass drifted to the left and half to the right. There continued to be à tendency for the centers of mass to rise as the range increased. As a matter of fact, all ten centers of mass rose as the range increased. In order to remove this

random drift, the data was again modified by subtracting the coordinates of the center of mass from the coordinates of each impact point in that round. The centers of mass for the smooth barrel data are provided in Table 2.

However, even with the data modified in this way, the values of the test statistic were unacceptably large. Only the Polar Gamma distribution produced a test statistic less than 100. While this student was preparing a histogram of values for the radius vector, he noticed that rounds 6, 7, and 8 produced an inordinate number of large values for the radius vector. This caused him to suspect that these rounds might be classified as outliers. Consequently, he calculated the sample variance for the three rounds in question and the sample variance for the remaining seven rounds. When the ratio of these sample variances was calculated, an F statistic was produced. The F statistic for the 10 meter data was 21.9, for the 30 meter data 30.3. The critical value for a level of significance of five percent and degrees of freedom 46,112 is approximately 1.55. Since the test statistics were large with respect to the critical value, this student classified rounds 6, 7, and 8 outliers and removed them from the data set. This variation among the rounds may have been produced by variation in the manufacture or performance of the ammunition. In any event, there seems to be sufficient reason to desire a repeat of this experiment to see if this variation is observed again.

TABLE II

CENTERS OF MASS-SMOOTH BARREL

Round	10 Meters	30 Meters
1	(-0.16,2.23)	(1.11,3.85)
2	(0.33,3.43)	(0.13,5.85)
3	(-0.98,2.40)	(-2.65,8.66)
4	(-1.73,2.58)	(-3.49,8.35)
5	(-0.30,2.53)	(-1.88,6.21)
9	(0.11,3.56)	(1.40,10.73)
10	(-1.09,3.32)	(-0.20,5.81)
Overall Mean	(-0.54,3.37)	(0.79,7.06)

If 30 regions had been used for the goodness-of-fit test after the data set had been reduced by the exclusion of the three outlying rounds, this would have resulted in an expected frequency of only 3.73, much less than the desired five. Consequently, the number of regions used when testing the fit of the Polar and Polar Gamma distributions was reduced to 24 by superimposing only three circles over the six rays from the origin. The radii corresponded to the estimated 25, 50, and 75 percentiles of the distribution of the radius vector. The use of 24 regions raised the expected frequency to 4.67 which is acceptable [Ref. 3].

After eliminating the atypical rounds, reducing the number of regions in the partition of the plane, and reestimating the gamma parameters based on the remaining seven rounds, the three distributions were tested for fit. The bivariate normal and Polar distributions resulted in relatively large test statistics; all were over 30. The Polar Gamma produced a test statistic PF 23.8 for parameter values of alpha equal to 4 and lambda equal to 1.8934, using the 10 meter data. The lowest test statistic for the 30 meter data was 24.7, obtained with the Polar Gamma distribution with parameter estimates of alpha equal to 4 and lambda equal to 0.5352. Since the smooth barrel data had been modified in the same manner as the rifled barrel data, the same problems were encountered. Since the Polar Gamma distribution with an alpha value of 4 produced the lowest value of the test statistic for both sets of data, it was recommended to model the behavior of the smooth barrel data.

As was the case with the rifled data, the Polar Gamma distribution was preferred over the Polar distribution because of the chacteristics of the gamma distribution, because the histogram of values for the radius vector appeared to be gamma, and because the Polar Gamma lent itself to modification as the range varied.

VI. RECOMMENDATIONS

The family of Polar Gamma distributions should be used to model the scatter of pellets of the XM261 round about the center of mass of each round. The parameters for the gamma distribution should be estimated from the data using the method of moments estimates.

The drift of the center mass as a function of range should be introduced into simulations of these rounds. The method used should depend on the amount of data available. See Appendix C for possibilities when data is limited.

The variation of the scale parameter lambda should be modeled as a function of range in an appropriate manner.

Additional data should be taken to allow a more complete investigation of the distribution of the center mass of a single round as a function of range. Additional data should also be taken to allow a more complete investigation of the variation of the scale parameter lambda as a function of range.

APPENDIX A

ACCURACY DATA

Smooth Bore Barrel

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 meters X(in.)	Y(in.)
1a	-0.1	+0.4	+3.2	-2.5
b	+0.4	+0.5	-1.5	-2.4
c	-1.6	+0.6	+0.8	-1.8
d	-1.7	+0.9	-3.7	+0.1
e	-0.8	+1.0	+5.8	+2.2
f	-0.4	+1.3	+2.0	+2.4
g	+1.1	+1.9	+1.4	+2.8
h	-0.3	+2.3	+0.4	+3.7
i	-1.0	+2.7	-2.8	+4.7
j	-0.8	+2.8	-1.4	+4.8
k	+2.4	+3.1	+8.1	+7.4
l	+2.0	+3.3	-2.1	+7.5
m	+1.7	+3.4	+10.5	+7.7
n	-3.9	+3.7	+6.4	+7.9
o	-0.1	+3.8	-11.8	+8.1
p	+0.4	+4.1	+2.6	+9.1
2a	+1.1	-0.1	+3.4	-3.7
b	0.4	+0.9	-1.2	-0.5
c	-0.1	+1.3	-3.1	-0.5
d	-0.6	+1.8	0.0	+0.3
e	+0.9	+2.6	+2.3	+2.9
f	-1.4	+2.7	-4.5	+3.9
g	+0.9	+2.7	+0.9	+5.4
h	+0.9	+3.4	+1.4	+5.5
i	+4.1	+3.9	-4.2	+7.4
j	+0.5	+4.0	+11.6	+7.8
k	+0.6	+4.4	+0.3	+8.0
l	-0.7	+4.4	+0.4	+8.7
m	-0.8	+4.7	-1.0	+10.2
n	+0.8	+5.6	+3.9	+10.8
o	+0.9	+5.9	+1.1	+10.9
p	-2.1	+6.7	-9.2	+16.6

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
3a	-2.0	-0.8	-5.8	-3.0
b	-1.1	+0.3	-4.5	+0.3
c	-1.5	+1.4	-3.6	+4.5
d	-0.6	+1.7	+5.3	+4.6
e	-0.2	+1.8	-2.8	+5.2
f	-0.6	+1.8	-0.7	+5.5
g	-0.6	+1.9	-1.9	+7.2
h	-0.7	+2.1	-1.0	+7.4
i	+2.1	+2.2	-1.0	+9.1
j	-2.5	+2.5	-3.2	+9.9
k	-0.5	+3.2	-1.4	+10.6
l	-1.2	+3.2	-7.5	+11.8
m	-2.4	+3.9	-7.2	+13.2
n	-2.9	+4.1	-8.5	+16.0
o	-3.0	+4.2	+5.6	+17.7
p	+1.9	+4.9	-4.2	+19.1
4a	-1.2	-0.7	-2.0	-4.2
b	-4.1	+0.4	-5.9	+1.2
c	-2.6	+0.5	-10.1	+1.6
d	-1.5	+1.4	-2.8	+4.9
e	+1.6	+1.7	+8.7	+5.2
f	-0.9	+2.0	-7.1	+6.7
g	-3.4	+2.1	-0.8	+7.6
h	-1.1	+2.4	-2.3	+8.1
i	+0.3	+2.7	-1.1	+8.3
j	-1.3	+3.1	+2.3	+9.1
k	-0.5	+3.4	+0.7	+12.2
l	-2.9	+3.6	-6.4	+12.3
m	-1.9	+3.8	-8.2	+12.6
n	3.1	+4.4	-9.6	+13.6
o	-3.9	+4.6	-8.1	+14.8
p	-1.2	+6.0	-3.2	+19.7

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
5a	+0.2	+0.2	-0.9	-1.9
b	+1.0	+0.7	+2.4	-1.7
c	+1.1	+0.8	-4.5	-0.1
d	+0.6	+1.0	+5.7	+0.1
e	-1.1	+1.1	+1.8	+0.6
f	-1.5	+1.5	-6.0	+2.7
g	+2.2	+1.7	+3.6	+3.1
h	+0.9	+1.9	-3.1	+4.9
i	-1.4	+2.5	+1.1	+5.9
j	-0.2	+2.6	-8.7	+7.6
k	-2.3	+3.0	-14.5	+11.7
l	+0.7	+4.1	-5.6	+12.1
m	-1.6	+4.7	-5.8	+12.5
n	-1.6	+4.8	+3.2	+12.5
o	-3.7	+4.8	+1.0	+14.6
p	+1.4	+5.1	+2.2	+14.9
6a	+2.6	-11.9	+7.1	-36.1
b	+2.0	-7.5	+6.3	-21.3
c	-1.1	-6.1	-8.8	-20.1
d	-1.3	-5.0	-8.5	-16.5
e	+6.8	-3.7	+18.7	-11.5
f	+8.8	-3.4	+23.4	-11.5
g	-6.2	+1.1	-18.0	+2.0
h	-3.2	+2.8	-5.4	+11.1
i	-1.4	+3.5	+10.4	+11.4
j	-1.5	+3.7	-4.1	+11.9
k	+3.2	+4.4	-10.2	+12.5
l	+3.5	+4.7	+8.0	+15.3
m	-3.5	+7.0	+3.6	+24.4
n	+1.5	+8.4	-8.5	+28.1
o	-2.6	+8.6	-9.7	+22.0
p	-1.5	+20.9	-18.4	+26.2

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
7a	-0.4	+0.7	+9.8	-1.1
b	+2.1	+1.1	-1.9	-0.5
c	+0.4	+2.2	+1.4	+0.3
d	+2.0	+2.3	+9.0	+3.0
e	-0.2	+2.4	+0.7	+3.7
f	+1.1	+2.6	+3.8	+3.8
g	-0.3	+2.8	+0.4	+4.2
h	-1.2	+3.2	-3.2	+6.5
i	-2.9	+3.5	-6.9	+6.6
j	+0.6	+3.7	+1.7	+7.9
k	-0.1	+3.9	+1.5	+7.3
l	0.0	+4.1	+1.9	+8.9
m	+1.8	+4.5	+8.1	+9.9
n	+1.2	+5.2	+13.0	+12.0
o	+1.4	+5.3	-1.8	+12.1
p	+3.4	+5.7	+1.9	+14.5
8a	+0.6	+8.3	+3.8	+26.1
b	+4.4	+1.9	+15.7	+5.5
c	+16.2	+0.0	+28.9	-14.1
d	+0.4	-4.8	+25.6	-14.0
e	+7.6	-3.9	+4.5	-17.0
f	+8.6	-4.1	+4.4	-23.6
g	+5.9	-7.0	+20.3	-24.0
h	+0.2	-7.6	+23.4	-40.8
i	+6.8	-12.1	-19.4	-4.6
j	+3.0	-10.2	-17.8	+13.7
k	+2.9	-10.3	+27.6	-15.3
l	-8.1	-1.1	+3.2	+6.7
m	-0.1	+15.5	+29.6	-5.8
n	-0.3	+15.9	-26.4	-37.4
o	*	*	*	*
p	*	*	*	*

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with sequence listed at 10 meters.

* Did not impact on target.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
9a	+4.3	-7.3	+16.8	+14.6
b	+1.8	-6.7	+22.8	+8.2
c	0.0	-7.0	+24.2	+2.3
d	-0.2	-7.1	+27.8	+1.0
e	-7.0	-2.0	+12.2	-24.5
f	-9.8	-1.1	+7.5	-20.9
g	-12.2	-1.0	0.0	-22.3
h	-12.3	+0.9	-3.4	-24.6
i	-4.7	+9.7	-21.4	-8.1
j	-4.7	+10.8	-29.7	-5.8
k	-5.1	+11.7	-38.5	-7.7
l	-6.2	+11.5	-37.3	0.0
m	-9.1	+18.3	-21.0	+32.2
n	+7.7	+2.9	-16.6	+31.9
o	+8.0	+1.7	-15.2	+29.6
p	+9.9	+1.1	-14.2	+35.4
10a	+7.1	-0.5	+1.3	+32.6
b	+6.7	-2.7	+3.0	+28.6
c	+8.4	+0.1	+17.8	+8.8
d	+6.3	+2.8	+24.7	+1.0
e	+2.3	-8.0	+21.4	-2.2
f	-0.5	-9.0	+21.1	-8.3
g	-2.8	-8.3	+7.0	-24.4
h	-4.8	-6.9	-1.6	-26.9
i	-8.3	-2.2	-7.6	-25.6
j	-10.6	-1.1	-14.3	-21.4
k	-11.0	+1.9	-24.0	-6.9
l	-9.2	+3.3	-31.8	-3.5
m	-4.7	+5.6	-33.6	+5.1
n	-1.4	+8.0	-28.3	+9.4
o	-0.5	+8.8	-15.4	+17.2
p	+1.3	+9.0	-5.0	+25.5

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

RIFLED BARREL

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
1a	+10.0	-6.9	+33.0	+12.5
b	+6.9	-9.1	+28.4	+14.8
c	-12.7	-10.0	+25.3	+26.5
d	+3.2	-8.5	+30.3	+22.5
e	-5.7	-7.1	+30.6	-17.4
f	-9.9	-5.6	+21.0	-26.2
g	-10.0	-5.7	+8.2	-25.6
h	-8.1	-2.1	+6.1	-32.8
i	-7.6	+9.7	-17.7	-23.1
j	-4.6	+12.6	-29.1	-17.8
k	-6.3	+13.5	-25.2	-7.6
l	-8.1	+13.1	-22.6	-32.4
m	+8.9	+8.6	-29.5	+37.1
n	+10.8	+7.0	-27.8	+29.0
o	+10.2	+5.1	-21.5	+38.5
p	+11.6	+3.7	-16.9	+38.0
2a	+6.6	-8.8	+21.5	+16.6
b	+5.6	-7.8	+19.8	+15.2
c	+5.8	-6.7	+21.2	+13.8
d	+2.8	-4.9	+20.9	+12.8
e	-5.7	-5.3	-17.7	-13.3
f	-7.7	-3.0	+15.7	-21.1
g	-10.5	-1.8	+21.8	-23.7
h	-11.5	-2.9	+21.4	-24.8
i	-3.1	+9.9	-18.0	-16.8
j	-4.1	+8.3	-23.7	-9.7
k	-4.6	+7.8	-31.5	-9.0
l	-6.1	+11.0	-34.8	-6.6
m	+7.2	+14.1	-20.5	+30.6
n	+8.1	+5.1	-17.9	+22.5
o	-17.9	+3.8	-13.4	+27.4
p	+7.7	+3.4	-10.6	+30.9

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
3a	+3.4	-7.4	+35.3	+6.3
b	-0.9	-8.6	+33.4	+33.5
c	-1.8	-8.1	+33.0	+33.4
d	-3.7	-9.5	+18.3	+36.5
e	-7.4	+1.8	+8.1	+38.4
f	-6.5	+3.2	+10.3	+4.8
g	-11.3	+1.8	+2.7	+34.3
h	-12.6	+3.7	-1.0	+40.0
i	-5.2	+5.4	-21.4	+5.6
j	-3.6	+13.1	-23.1	+2.5
k	+0.9	+13.8	-37.2	+1.3
l	+1.8	+15.7	-40.2	+7.2
m	+2.2	+15.7	-2.7	-28.5
n	+4.0	+1.9	-4.9	-27.0
o	+12.3	+3.0	-11.6	-31.6
p	+14.6	+4.3	+9.1	-23.1
4a	+12.4	-7.0	+29.5	+32.4
b	+12.2	-6.6	+24.0	+35.7
c	+7.0	-10.0	+21.5	+29.5
d	+7.5	-4.6	+16.2	+15.9
e	-4.0	-9.4	+38.4	-19.0
f	-6.4	-5.9	+35.1	-16.0
g	-8.0	-6.6	+21.0	-12.3
h	-10.9	-7.5	+6.1	-10.1
i	-9.0	+7.4	-7.5	-32.3
j	-9.9	+9.8	-25.2	-23.8
k	-4.7	+12.2	-38.8	-26.5
l	-13.3	+14.4	-30.3	-39.7
m	+8.4	+11.1	-31.2	+21.6
n	+8.0	+9.5	-33.5	+28.6
o	+10.3	+10.3	-33.5	+26.2
p	+7.1	+4.7	-38.4	+35.1

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
5a	+12.8	-13.6	+12.0	+17.2
b	+7.6	-4.1	+2.5	+24.0
c	+1.4	-2.5	+18.6	+0.1
d	+7.3	+0.2	+14.0	+21.5
e	-5.0	-11.5	+2.5	-6.7
f	-11.7	-15.6	+22.5	-9.5
g	-4.8	-8.3	+40.2	-39.5
h	-15.3	-5.9	+13.8	-35.5
i	-14.8	-2.0	-4.4	-33.5
j	-16.7	-2.2	-14.2	-37.0
k	-12.3	+1.5	-14.1	-24.0
l	-0.1	+13.0	-35.8	-47.5
m	+4.2	+15.3	-42.0	+7.7
n	+3.6	+7.3	-30.5	+32.5
o	+6.0	+5.4	-26.7	+30.2
p	+6.0	+2.0	-23.6	+31.2
6a	+11.5	+1.5	+0.8	+34.6
b	+9.5	-0.9	+2.4	+37.6
c	+7.9	+0.3	+23.8	+2.9
d	+7.9	-0.8	+35.8	+1.1
e	-2.9	-9.2	+24.7	-5.0
f	-3.9	-8.4	+28.5	-5.6
g	-2.6	-6.5	+4.2	-31.9
h	+1.7	-9.6	-7.5	-21.2
i	-10.8	+1.7	-10.0	-30.0
j	-9.5	+4.0	-12.4	-29.3
k	-12.4	+4.9	-31.6	+2.7
l	-13.9	+4.9	-31.1	+9.2
m	+0.5	+12.3	-38.5	+12.1
n	+1.4	+13.0	-44.6	+12.2
o	+3.2	+14.8	-44.3	+42.0
p	+5.5	+15.3	-35.1	+32.0

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
7a	+9.5	-3.0	+1.6	+32.6
b	+6.0	-3.0	+6.5	+38.8
c	+4.2	-3.8	+11.0	+41.5
d	+0.9	-10.3	+29.5	+1.9
e	-3.1	-9.2	+28.1	-8.9
f	-4.0	-9.3	+17.0	-9.5
g	-4.5	-8.4	+11.0	-9.4
h	-9.7	+1.0	+24.6	-34.7
i	-11.0	+2.8	+2.6	-30.0
j	-10.9	+3.9	-15.8	-24.0
k	-7.7	+3.5	-12.0	-25.9
l	-1.3	+9.8	-29.8	+3.1
m	+1.4	+10.1	-33.8	+8.4
n	+2.6	+12.3	-36.0	+10.4
o	+3.9	+13.4	-23.8	+9.9
p	+10.1	+0.5	-4.0	+30.2
8a	+7.0	-4.7	+23.3	+20.3
b	+6.2	-7.3	+14.6	+20.0
c	+3.7	-7.0	+16.0	+16.5
d	+1.0	-7.4	+15.6	+13.0
e	-6.2	-5.1	+21.0	-14.8
f	-7.8	-3.2	+19.9	-23.6
g	-7.9	-2.5	+13.3	-22.0
h	-7.9	-2.3	+3.4	-25.5
i	-4.6	+8.9	-19.6	-19.1
j	-5.0	+9.2	-23.4	-14.6
k	-5.3	+9.7	-20.5	-9.0
l	-5.9	+10.2	-211	-2.0
m	+5.1	+7.1	-18.0	+30.6
n	+5.7	+5.8	-16.0	+28.0
o	+5.6	+5.5	-13.7	+24.3
p	+7.8	+6.8	-9.1	+18.1

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with sequence listed at 10 meters.

Shot Number	Impact at 10 Meters X(in.)	Y(in.)	Impact at 30 Meters X(in.)	Y(in.)
9a	+0.9	+1.7	+3.7	+4.4
b	-1.9	+1.9	-7.1	+5.1
c	-0.8	+2.8	+8.4	+7.8
d	-4.6	+2.8	-2.8	+8.6
e	-1.4	+2.9	-13.8	+8.8
f	+2.2	+3.0	+7.9	+8.7
g	+0.4	+3.1	+8.2	+9.0
h	+2.3	+3.2	+7.2	+9.5
i	+2.0	+3.3	-3.7	+9.7
j	+2.1	+3.4	+4.9	+10.3
k	+1.2	+3.5	+3.2	+11.6
l	-1.3	+4.2	+0.4	+11.9
m	-0.2	+4.3	-4.5	+12.5
n	-2.1	+5.2	-4.3	+15.7
o	+0.8	+5.7	+6.2	+17.7
p	+2.3	+6.1	+8.6	+20.4
10a	-0.5	-1.1	+1.6	-9.4
b	-0.4	+0.3	+2.8	-3.7
c	0.0	+1.9	+2.8	+1.5
d	-2.3	+2.7	-1.8	+3.7
e	-1.0	+2.8	+7.3	+4.4
f	+0.8	+2.9	-3.7	+4.6
g	-1.5	+3.0	+6.9	+4.8
h	-2.1	+3.1	-3.4	+4.9
i	-1.8	+3.4	-4.5	+5.1
j	-1.6	+3.8	+1.6	+6.2
k	-1.3	+3.9	-1.9	+6.5
l	+1.7	+4.2	-3.3	+9.8
m	-0.4	+4.8	+2.7	+9.9
n	-3.5	+5.6	-4.7	+14.2
o	-2.2	+5.8	-3.7	+14.3
p	-1.4	+6.1	-1.9	+16.3

Sequence of sub-projectiles listed at 30 meters is not necessarily identical with the sequence listed at 10 meters.

APPENDIX B

PARAMETER ESTIMATION BY THE METHOD OF MOMENTS

$$X_i \sim \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \text{ for each } i = 1,\ldots,n \ x_i \text{ iid}$$

The expected value of a gamma distributed random variable is α/λ, the variance is α/λ^2. To use the method of moments set the sample mean equal to the theoretical mean and the sample variance equal to the theoretical variance. Thus

$$\bar{x} = \alpha/\lambda \text{ and } s^2 = \alpha/\lambda^2$$

Substituting $\bar{x}$ for its equivalent expression in the relationship between s^2 and the distribution parameters yield:

$$s^2 = \frac{1}{\lambda}(\delta) \text{ or } \lambda = \frac{\bar{x}}{s^2}$$

Substituting the value for lambda in the expression for $\bar{x}$ yields: $\bar{x} = \dfrac{\alpha}{\frac{\bar{x}}{s^2}}$ or $\alpha = \dfrac{\bar{x}^2}{s^2}$.

Therefore, the method of moments produces the following estimates for the parameters:

$$\hat{\alpha} = \frac{\bar{x}^2}{s^2} \quad \hat{\lambda} = \frac{\bar{x}}{s^2}$$

Since $\hat{\alpha}$ and $\hat{\lambda}$ both depend on $\bar{x}$ and s^2, they are positively correlated.

The bias b (θ) of an estimate is defined as b (θ) = E (T) - θ where T is the statistic used to estimate θ, the true parameter value. Approximate the bias by using a first order Taylor expansion:

$$\hat{\lambda} = \frac{\bar{x}}{s^2} \approx \frac{\bar{x}}{\sigma^2} - \frac{1}{\sigma^4}\,\bar{x}\,(s^2-\sigma^2)$$

$$E(\hat{\lambda}) \approx E(\frac{\bar{x}}{\sigma^2}) - E[\frac{\bar{x}}{\sigma^4}\,(s^2-\sigma^2)]$$

$$\frac{\mu}{\sigma^2}\;(\frac{\lambda^3}{\sigma^4}\;\text{cov}\;(s^2,\bar{x})$$

$$\frac{\alpha}{\lambda}\;(\frac{\lambda^3}{\alpha}) - \frac{1}{\sigma^4}\;\text{cov}\;(s^2,\bar{x})$$

$$\lambda - \frac{1}{\sigma^4}\;\text{cov}\;(s^2,\bar{x})$$

$E(\hat{\lambda}) = \lambda - \frac{1}{\sigma^4}\;\text{cov}\;(s^2,\;\bar{x}) < \lambda$ since s^2 and $\bar{x}$ are positively correlated.

$\hat{\alpha} = \frac{\bar{x}^2}{s^2}$ using the Taylor expansion for $s^2 \approx \frac{\bar{x}^2}{\sigma^2} - \frac{\bar{x}}{\sigma^4}\,(s^2-\sigma^2)$

$$E(\hat{\alpha}) = E(\frac{\bar{x}^2}{\sigma^2}) - E[\frac{\bar{x}}{\sigma^4}(s^2-\sigma^2)]$$

$$\frac{\mu^2}{\sigma^2} - \frac{1}{\sigma^4}\;\text{cov}\;(\bar{x}^2,s^2)$$

$$\frac{\alpha^2}{\lambda^2}\;\frac{\lambda^2}{\alpha} - \frac{1}{\sigma^4}\;\text{cov}\;(\bar{x}^2,s^2)$$

$E(\hat{\alpha}) = \alpha - \frac{1}{\sigma^4}\;\text{cov}\;(\bar{x}^2,\;s^2) < \alpha$ since $\bar{x}^2$ s^2 are positively correlated

APPENDIX C

CENTER OF MASS AS A FUNCTION OF RANGE

The information about the center of mass of any one round is limited. The location of the center of mass is known at three ranges: 0, 10, and 30 meters. At zero range the center of mass is at the origin since the pellets would not have had an opportunity to disperse. The location at 10 and 30 meters can be calculated from the data. With this limited information it is difficult to accurately depict the motion of the center of mass. Additional experimentation to provide more data on the motion of the center of mass would allow a more thorough examination of the motion of the center of mass as a function of range. A desirable situation would be to have sufficient data to allow an analyst to determine a distribution of the center of mass. In the absence of further information, the data at hand should be used. Therefore, two possible approaches to model the motion of the center of mass are presented here. Users of the results of this study should determine what procedure best satisfies their view of the drift of the center of mass.

One method to model the drift of the center of mass is for each one of the coordinates to fit the lowest degree polynomial that appears reasonable. An examination of the data would seem to eliminate a constant or a linear relationship between either coordinate of the center of mass and range. Since a parabola contains three parameters and coordinate

values are available at three ranges for each shot fired, it is possible to fit a parabola through the coordinate values for each coordinate of each center of mass. Using the parabolas determined for the two coordinates of a center of mass, the position of the center of mass can be determined for any range. However, the procedure does not provide any insight into the physical phenomenon producing the motion. It does provide a method of modeling the motion of the center of mass as a function of range which will reproduce the experimental data.

A model of the drift of the center of mass can be based on the physics of motion if one can accept three assumptions: the forces acting on the projectile (the recoil and air resistance) act for only a short period of time, once the covering on the pellets is aerodynamically stripped away air resistance is negligible, and gravity does not significantly affect the pellets over their maximum effective range. Under these assumptions, the forces causing the center of mass to drift can be visualized as acting instantaneously as the projectile leaves the barrel. After leaving the barrel, the projectile would travel in a straight line. One can then use the two data points at 10 and 30 meters to estimate the parameter of a straight line through the origin using the least squares method. The line must go through the origin because at zero range the center of mass is at the origin.

This model has the appeal of being based on physical principles of motion. However, it has the disadvantage of not reproducing the actual experimental results because it is unlikely that a least squares fitted line will go through either of the data points.

Once the user has selected a method of modeling the motion of the center of mass as a function of range, he can use that model to generate a function for each of the experimental rounds. When a round is "fired" in a simulation of one of these functions based on experimental data can be randomly selected to provide the position of the center of mass at the range desired for the simulation. In this manner the drift of the center of mass can be reintroduced to the simulation.

APPENDIX D

CHANGE IN DISTRIBUTION DUE TO CENTERING

Since it was convenient to modify the original data by subtracting the coordinates of the center of mass of each round from the coordinates of each pellet in the round, it became necessary to explore the effect that this modification would have on the distribution of the pellet strikes. Although the bivariate normal did not adequately describe the data, it was sufficiently tractable to allow an analytical examination of the change in distribution due to centering at the center of mass. Since the purpose of this examination is to determine the general characteristics of the change produced, it seems reasonable to use the bivariate normal. Then one can use the intuition produced by this examination to understand what is occurring in less tractable distribution.

Therefore, the distribution used for the examination was the bivariate normal with independent variables. Without loss of generality and for the sake of convenience the mean for both variables was assumed to be zero and the variance one.

$\{(X_i, Y_i)\}$ is a random sample of size N from the X, Y distribution with

$(X,Y) \sim N\,(\underline{O},\ I_2)$

$(\bar{X},\ \bar{Y}) \sim N\,(\underline{O}, \frac{1}{N}\ I_2)$

$(X-\bar{X}, Y-\bar{Y}) \sim N\,(\underline{O}, V)$ where the variance matrix V is diagonal and has the form:

$$\begin{matrix} VARX + VAR\bar{X} - 2COV\,(X,\bar{X}) & 0 \\ 0 & VARY+VAR\bar{Y}-2COV\,(Y,\bar{Y}) \end{matrix}$$

Since the distribution of X and Y are identical, it is possible to determine the entries in the V matrix by evaluating the COV $(X,\bar{X})$ term.

$$COV\,(X_i,\bar{X}) = COV\,(X_i, \frac{1}{N}\Sigma X_i)$$

Since the covariance is a linear operator, the constant and the sum can be brought outside the covariance operator.

$$COV\,(X_i, \frac{1}{N}\Sigma X_i) = \frac{1}{N}\Sigma\ COV(X_i, X_j) = \frac{1}{N}\Sigma\,\delta_{ij} = \frac{1}{N}$$

Therefore, the V matrix has elements $\begin{matrix} 1-\frac{1}{N} & 0 \\ 0 & 1-\frac{1}{N} \end{matrix}$

So $(X-\bar{X}, Y-\bar{Y}) \sim N(\underline{O}, (\frac{N-1}{N})\ I_2)$

The square of the distance from the center of mass to a point with modified coordinates is:

$$D_M^2 = (X-\bar{X})^2 + (Y-\bar{Y})^2 \quad \text{or} \quad \frac{D_M^{\ 2}}{(\frac{N-1}{N})} = \frac{(X-\bar{X})^2}{(\frac{N-1}{N})} + \frac{(Y-\bar{Y})^2}{(\frac{N-1}{N})}$$

Each term on the right side of the last equality will be recognized as the square of a standard normal. Hence

$$\frac{D_M^{\ 2}}{(\frac{N-1}{N})} \sim \chi^2_{(2)}.$$

The square of the distance from the origin to one of the points with unmodified coordinates is: $D^2 = X^2 + Y^2$. Since X and Y are standard normals $D^2 \sim \chi^2_{(2)}$.

Since $\dfrac{D_M^2}{(\frac{N-1}{N})}$ and D^2 have the same probability distribution, it follows that for any real $K>0$ that $P\left(\dfrac{D_M^2}{(\frac{N-1}{N})} < K\right) = P(D^2 < K)$ or

$$P(D_M^2 < (\frac{N-1}{N})K) = P(D^2 < K)$$

Since $(\frac{N-1}{N})K < K$ for all integer $N>0$, the above probability statement means that C.D.F. for D_M^2 lies above the c.d.f. for D^2 for all $K>0$. Hence the probability that a point with modified coordinates lies within a given distance from the origin (i.e., the center of mass for that group of pellets) is greater than the probability that an original point lies within that same distance from the origin. Therefore, the modification of the data diminishes the effect of an area of low probability at the origin.

It should be remembered that the above discussion concerns the distributional characteristics of the pellet strikes and assumes that all groups of pellets have the same origin. It is under these circumstances that the "hole" in the center of the data is reduced. When taking experimental data, the different groups of pellets don't actually have the same expected value for their centers of mass. And, as was mentioned in the body of the thesis, when these groups are made to have their centers of mass coincide, a "hole" may be produced.

APPENDIX E

SIMULATION TO DETERMINED THE CRITICAL VALUES FOR THE GOODNESS-OF-FIT TEST

The data generated by the computer was designed to closely approximate the characteristics of the experimental data. Although the characteristics of the computer data did not exactly duplicate the characteristics of the experimental data, the deviation was not significant. Since the Polar Gamma distribution had produced the lowest value for the goodness-of-fit test statistic for all sets of data, the simulation was based on a Polar Gamma distribution.

During the simulation, sets of data were produced which corresponded to the data for the Polar Gamma distribution. One hundred sixty pairs of uniform and gamma random variables were produced. A pair of X and Y variables was then calculated from each pair of uniform and gamma variables using the relationships: $X = \rho \cos \theta$ and $Y = \rho \sin \theta$. This point in the simulation corresponds to the point immediately after the data modification in the actual procedure. In both cases there exists pairs of X and Y data which would produce uniformly distributed angles and gamma distributed radius vectors. The gamma parameters were then estimated using the method of moments as derived in Appendix C. The values for the radii of the circles were then read from a vector. The values selected were based on the value of the alpha parameter produced by truncating the estimated value at the decimal point.

At this point it should be noted that the entire simulation is independent of the value for lambda. Using the relationship between the gamma distribution with an integer value for alpha and the Poisson distribution, the values for $\lambda\rho$ corresponding to a given alpha value were interpolated from tables of cumulative Poisson elements. These table values had to be scaled by a factor of $1/\lambda$ to produce the ρ value for the percentile of the gamma distribution.

A formula for generating a gamma random variable is $\rho = -1/\lambda \ln (\prod_{i=1}^{\alpha} R_i)$ where R_i is distributed uniformly on the unit interval. Therefore, the values of ρ are also scaled by a factor of $1/\lambda$. (The minus sign only compensates for the logarithm of a number less than one being negative.) Hence, both the value for the gamma random variable and the values for the percentiles of the gamma variable are scaled by the same factor. This is equivalent to saying that the scaling factor produces no effect on the result of the simulation. Therefore, the critical values produced by the simulation are only a function of the alpha value.

After the radii of the test circles were determined, a goodness-of-fit statistic was generated. Then a second set of radii were determined using the truncated value of alpha plus one. A second goodness-of-fit statistic was generated and compared to the first. The smaller of the two statistics was stored.

After 1000 runs through the program , the stored test statistics were sorted from smallest to largest. The 5, 10, 25, 50, 75, 90, and 95 sample percentiles were printed. The 90 and 95 sample percentiles were used to represent the critical values for this goodness-of-fit test. The sample percentiles are given in Table 3 as a function of the alpha value.

TABLE IV

SAMPLE PERCENTILES FROM GOODNESS-OF-FIT SIMULATION

	SAMPLE PERCENTILE						
ALPHA	5	10	25	50	75	90	95
3	21.8	24.5	33.1	84.5	157.2	185.3	199.2
4	26.0	31.2	42.1	65.3	105.1	135.5	162.8
5	20.0	22.2	28.6	36.8	45.8	56.0	65.3
6	20.0	23.0	28.6	36.1	46.2	57.5	65.7
7	21.8	24.5	30.1	41.0	54.8	68.0	75.1
8	21.5	24.5	30.1	39.5	55.6	69.5	76.2
9	19.6	22.2	27.8	34.2	43.2	54.5	63.5
10	19.6	21.5	26.3	32.0	38.7	45.8	50.7
11	19.6	21.5	26.0	31.6	36.8	42.8	46.6
12	19.2	21.8	25.6	31.2	37.2	42.5	46.6
13	18.5	21.1	25.2	30.8	36.5	42.5	47.0
14	19.2	21.5	26.0	30.5	36.1	41.7	45.1
15	19.2	21.1	24.8	30.1	38.7	41.3	46.2
16	18.5	20.7	24.8	29.3	35.3	41.3	44.7
17	18.5	20.7	24.5	30.1	35.7	41.7	45.1
18	17.7	20.0	24.5	30.1	35.3	41.0	44.7
19	19.2	21.5	25.2	30.5	36.1	42.5	45.8
20	18.8	20.7	25.6	30.1	36.5	41.7	45.1

APPENDIX F

VARIATION OF LAMBDA AS A FUNCTION OF RANGE

Since data for only two ranges was analyzed, only two values of lambda are available at different ranges. An additional value for lambda can be deduced by observing that the variance of a gamma random variable is α/λ^2 and the variance of the pellet strikes at zero range is essentially zero. Since α is a constant in the models of interest, λ must increase without bound as the range decreases to zero. This is the only way to produce a zero variance at zero range within this family of distributions.

Since lambda is unbounded, it is more convenient to determine the reciprocal of lambda as a function of range and require that function to go through the origin. This situation bears a strong resemblence to the problem of determining the drift of the center of mass as a function of range. As in that case, it would have been desirable to have additional information about the way lambda varied as range increased. If data had been taken every five or ten meters, a more detailed analysis of the variation of lambda could have been accomplished.

A parabola can be used to fit the three values of the reciprocal of lambda. This method has the advantage of exactly duplicating the experimental results at the known ranges. However, it offers no insight into the reason why lambda should be expected to vary in this way.

An alternative method can be based on the dimensionality of the reciprocal of lambda and the moments of the gamma distribution. Since lambda varies with the range and the reciprocal of lambda times a constant is the expected value of the range, then the reciprocal of lambda must have dimensions of length. Since the dimensions of the reciprocal of lambda and range are the same, this suggests that a linear relationship between the reciprocal of lambda and range might be appropriate. Therefore, it may be reasonable to use the least squares technique to estimate the slope of a line through the origin and use that line to determine the reciprocal of lambda, and hence lambda, as a function of the range.

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This paper is an analysis of the behavior of density shot XM261 when fired from the .45 caliber pistol (M1911A1). Density shot ammunition is a multiprojectile round which contains 16 small pellets. Both the standard rifled barrel and a smooth barrel were used to collect data on the behavior of the XM261 round. A family of distributions is selected for each barrel type to model the scatter of pellets from the XM261 round at varying ranges.

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